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Asymptotic Stability without Uniform Stability: Almost Periodic Coefficients

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It is a direct consequence of Floquet theory that in the case of linear systems of differential equations with periodic coefficients, stability of the zero solution implies uniform stability, and asymptotic stability implies uniform exponential asymptotic stability. W. Hahn [2, p. 202] has asked what similar theorems might hold for the almost periodic case.

The aim of this note is to prove by example that

THEOREM. *There exist linear equations with almost periodic coefficients for which the zero solution is asymptotically stable, but not uniformly stable.*

In particular results such as those stated above cannot hold in general for almost periodic systems.

For the example, we consider a one-dimensional system,

$$\dot{x} = -f(t)\,x,$$

and its solutions

$$\varphi(t, x_0, t_0) = x_0 \exp\left(-\int_{t_0}^{t} f\right),$$

where f is a uniformly almost periodic function.

Denoting the mean value of f by

$$m = \lim_{t\to\infty} \frac{1}{t}\int_0^t f,$$

we observe that if $\int_0^t (f-m)$ is bounded in t, then each solution is the product of a uniformly almost periodic function, namely, $\exp\left(-\int_{t_0}^t (f-m)\right)$, and the exponential e^{-mt}. Therefore in this special case the periodic result does hold. However the integral of an almost periodic function need not behave so well.

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We give here a simple example of a uniformly almost periodic function, $f(t)$, such that

(1) $\int_0^t f \to \infty$ as $t \to \infty$.

(2) For any $M > 0$, t_0 and $t_1 > t_0$ can be found so that $\int_{t_0}^{t_1} f < -M$. (It follows from (1) and (2) that the mean value of f is zero.)

By the explicit representation given above for solutions of the differential equation, the existence of such an f implies the theorem.

Our function f will be expressed as

$$f(t) = \sum_{n=1}^{\infty} g_n(t)$$

where each g_n is a continuous periodic function of period 2^n which is zero on the first half of its period i.e.,

$$g_n(t) = 0 \qquad \text{for} \qquad 0 \leqslant t \leqslant 2^{n-1}.$$

If we let

$$m_n = \frac{1}{2^n} \int_0^{2^n} g_n$$

be the mean value of g_n, and introduce

$$a_n = \sum_{j=1}^{n} m_j ,$$

the above property of the g's allows us to compute:

$$\int_0^{2^n} f = 2^n a_n \qquad \int_{2^n}^{2^{n+1}} f = 2^n(2a_{n+1} - a_n).$$

Our first condition on f requires that $2^n a_n \to \infty$; and our second condition will be satisfied if the sequence, $2^n(2a_{n+1} - a_n)$, admits $-\infty$ as a limit point. With this in mind we collect some obvious facts into the following:

LEMMA 1. *Given a sequence a_n, we can choose functions g_n of period 2^n which are zero on the first half of their period* ($0 \leqslant t \leqslant 2^{n-1}$), *which have mean value $m_n = a_n - a_{n-1}$, and which satisfy*

$$\sup | g_n | < A | m_n |$$

where A is any preassigned constant which is greater than 2. (Note that A can be chosen as close to 2 as we like.) In particular, if $a_n = a_{n-1}$, then $g_n \equiv 0$.

Furthermore, if $\Sigma \mid a_n - a_{n-1} \mid < \infty$, then the function $f = \Sigma_{n=1}^{\infty} g_n$ is uniformly almost periodic.

Proof. Only the last statement needs comment and that follows since the condition implies $\Sigma \sup \mid g_n \mid < \infty$ so that f is the uniform limit of periodic functions.

We now choose a sequence a_n as follows:

(1) $a_1 = 1$.

(2) $a_n = a_{n-1}$ if n is even.

(3) $a_n = Ba_{n-1}$ if n is odd where $B < \frac{1}{2}$, $B^{1/2} > \frac{1}{2}$.

For this sequence we easily verify that

$$2^n a_n \geqslant 2^n B^{n/2} \to \infty,$$

and for n even,

$$2^n(2a_{n+1} - a_n) = 2^n(2B - 1)\, a_n \to -\infty.$$

Also we see that $\Sigma \mid a_n - a_{n-1} \mid < \infty$.

Thus this sequence will provide us with the desired example if we can prove the following lemma:

LEMMA 2. *If the constants A and B are chosen so that $A(1 - B) < \frac{3}{2}$, and if g_1 is chosen to be nonnegative, then the f constructed using Lemma 1 satisfies*

$$\int_0^t f \to \infty \qquad \text{as} \qquad t \to \infty.$$

Proof. By our choice of g_1 we can assume inductively that

$$\int_0^t f \geqslant 0 \qquad \text{for} \qquad t \leqslant 2^n.$$

Now let $2^n \leqslant t \leqslant 2^{n+1}$. Using the periodicity of the g's, we can write

$$\int_0^t f = \int_0^{2^n} f + \int_{2^n}^t f = 2^n a_n + \int_0^{t-2^n} f + \int_{2^n}^t g_{n+1}\,.$$

If n is odd, we have

$$m_{n+1} = a_{n+1} - a_n = 0 \qquad \text{and so} \qquad g_{n+1} \equiv 0.$$

An application of the induction hypothesis gives

$$\int_0^t f \geqslant 2^n a_n \to \infty. \tag{1}$$

If n is even, let r satisfy

$$r2^{n-1} \leqslant t - 2^n < (r + 1)\, 2^{n-1} \qquad (r = 0 \text{ or } 1).$$

We then have ($a_{n-1} = a_n$)

$$\int_0^{t-2^n} f \geqslant r\, 2^{n-1} a_{n-1} = r\, 2^{n-1} a_n$$

and

$$\int_{2^n}^{t} g_{n+1} \geqslant -(r + 1)\, 2^{n-1} \sup | g_{n+1} | \geqslant -(r + 1)\, 2^{n-1} A \mid m_{n+1} \mid .$$

Replacing $| m_{n+1} |$ by $a_n - a_{n+1} = (1 - B)\, a_n$, we have

$$\int_0^t f \geqslant 2^n a_n + r\, 2^{n-1} a_n - A(1 - B)(r + 1)\, a_n$$

$$= 2^{n-1}\{2 + r - A(1 - B)(r + 1)\}\, a_n . \tag{2}$$

By our choice of A and B, the expression in brackets is positive for $r = 0$ or 1. Thus we have verified the inductive hypothesis, and the lemma follows from the inequalities (1) and (2).

References

1. Besicovitch, A. S. "Almost Periodic Functions." Dover, New York, 1954.
2. Hahn, W. The present state of Lyapunov's direct method. *In* "Nonlinear Problems," R. Langer, ed. Univ. of Wisconsin Press, 1963.